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Heat Conduction from Two Spheres

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An exact solution of heat conduction from two spheres is obtained using the bispherical coordinates system. The two spheres may be of different diameters and may be located at any distance from each other. The solution is given in terms of the temperature distribution and the local and average Nusselt numbers. *AIChE J,* 56: 2248–2256, 2010
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Introduction

The problem of heat transfer from a single sphere has been investigated through many experimental and theoretical studies. On natural convection, the reader is referred to the work of Potter and Riley,[1] Brown and Simpson,[2] Geoola and Cornish,[3,4] Singh and Hasan,[5] Riley,[6] and Dudek et al.[7] The classic references related to forced and mixed convection past a single sphere are those by Dennis and Walker,[8] Whitaker,[9] Dennis et al.,[10] Sayegh and Gauvin,[11] Hieber and Gebhart,[12] Acrivos,[13] Wong et al.,[14] and Nguyen et al.[15] Studies related to heat or mass transfer from a sphere in an oscillating free stream are represented by Drummond and Lyman,[16] Ha and Yavuzkurt,[17] Alassar et al.,[18] and Leung and Baroth.[19]

Several important applications require the solution of the equation of heat conduction from two spheres. Some examples are the heat transfer in stationary packed beds, dispersed particles under the influence of a spatially uniform electric field (electrophoresis), and the thermo capillary motion of two spheres created by surface tension that develops when a temperature gradient is present around the bubbles. When the motion of the flowing fluid becomes slow, the solution of heat conduction represents a limiting case for forced convective heat/mass transfer around two spheres. In general, the results of this work can be of use in applications where Biot and Rayleigh numbers are small and fluid heat conduction dominates the thermal resistance. Such conditions can be found at small length scales. Furthermore, the impact of pure conduction is sometimes required for the effect of, for example, buoyancy and other forces to be isolated and studied. All the results in the work of Ha and Yavuzkurt[17] are presented in terms of $N_u - 2$. The value of 2 is the Nusselt number corresponding to pure heat conduction.

Heat conduction from a single sphere is a textbook problem. This simple problem was generalized by Alassar,[20] who investigated the conduction heat transfer from spheroids by solving the steady version of the energy equation subject to appropriate boundary conditions and showed that the solution for the sphere case can be obtained from his generalized results. Solomentsev et al.[21] found an asymptotic solution of Laplace equation over two equal nonconducting spheres of equal size when some field is applied perpendicular to the line of centers. Stoy[22] developed a solution procedure for the Laplace equation in bispherical coordinates for the flow past two spheres in a uniform external field. Dealing with Neumann boundary conditions, the determination of the coefficients of the orthogonal expansion is the central part of the work. It is interesting to know that the theoretically more complicated problem of convective heat transfer has been investigated; see for example, Juncu.[23] Juncu[23] numerically studied forced convective heat/mass transfer from two spheres that have the same initial temperature. Thau et al.[24] numerically solved Navier-Stokes and energy equations for a pair of spheres in tandem at $Re = 40$ for two different spacing using bispherical coordinates. Koromyslov and Grigor'ev[25] investigated an electrostatic interaction between two separate grounded uncharged perfectly conducting spheres of different radii in a uniform electrostatic field. Umemura et al.[26,27] investigated the effect of the interaction of two burning identical spherical droplets with the same radius[26] and different sizes[27] of the same kind of fuel. They obtained

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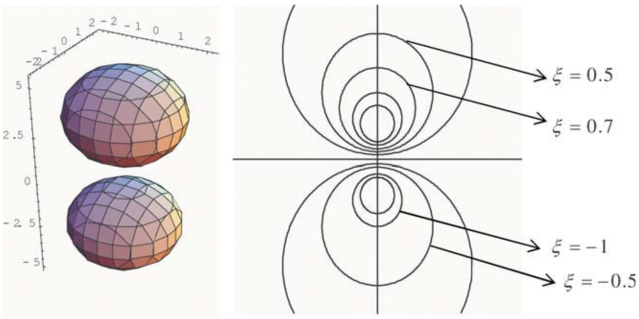

Figure 1. Surfaces of constant ξ.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

the burning rate and the form of flame surface in the two-droplet case. Brzustowski et al.[28] found the burning rate of two interacting burning spherical droplets of arbitrary size, when the mass fraction of the diffusing fluid vapor at the droplets surfaces is the same.

In this study, a simple exact solution of heat conduction from two isothermal spheres is obtained. The unconventional bispherical coordinates system is used to solve the problem. The two spheres may be of different diameters and different temperatures, and may be located at any distance from each other. The necessity and importance of considering two spheres may be summarized by the following statement by Cornish.[29] "It is well known that the minimum possible rate of heat (or mass) transfer from a single sphere contained within an infinite stagnant medium corresponds to a Nusselt (or Sherwood) number of two. Frequently, however, in multi-particle situations such as fluidized beds, values of the Nusselt number less than two have been measured. A variety of reasons—such as backmixing—have been put forward to explain this apparent inconsistency. It does not seem to have been generally realized that for multi-particle situations the minimum theoretical value of the Nusselt number can be much less than two."

Bispherical Coordinates System

The bispherical coordinates system (θ, ξ, γ) is a three-dimensional orthogonal coordinate system that results from rotating the two-dimensional bipolar coordinate system about the axis that connects the two foci. The two foci located at $(0,0,\mp a)$, under this rotation, remain as points in the bispherical coordinates system.[30]

The transformation equations of the bispherical coordinates system are

$$x = \frac{a\sin\theta\cos\gamma}{\cosh\xi - \cos\theta}, y = \frac{a\sin\theta\sin\gamma}{\cosh\xi - \cos\theta}, \text{and } z = \frac{a\sinh\xi}{\cosh\xi - \cos\theta} \tag{1}$$

The coordinates surfaces are

(a) Surfaces of constant $\xi(-\infty < \xi < \infty)$ given by

$$x^2 + y^2 + (z - a\coth\xi)^2 = \frac{a^2}{\sinh^2\xi} \tag{2}$$

which are nonintersecting spheres with centers at $(0,0,a\coth\xi)$ and radii of $\left|\frac{a}{\sinh\xi}\right|$ that surround the foci, Figure 1.

(b) Surfaces of constant θ ($0 \le \theta \le \pi$) given by

$$x^2 + y^2 + z^2 - 2a\sqrt{x^2 + y^2}\cot\theta = a^2 \tag{3}$$

which look like apples when $(0<\theta<\frac{\pi}{2})$, spheres when$(\theta = \frac{\pi}{2})$, and lemons when $(\frac{\pi}{2}<\theta<\pi)$, Figure 2.

(c) Surfaces of constant γ ($0 \le \gamma < 2\pi$) given by

$$\tan\gamma = \frac{y}{x} \tag{4}$$

which are half planes through the z-axis.

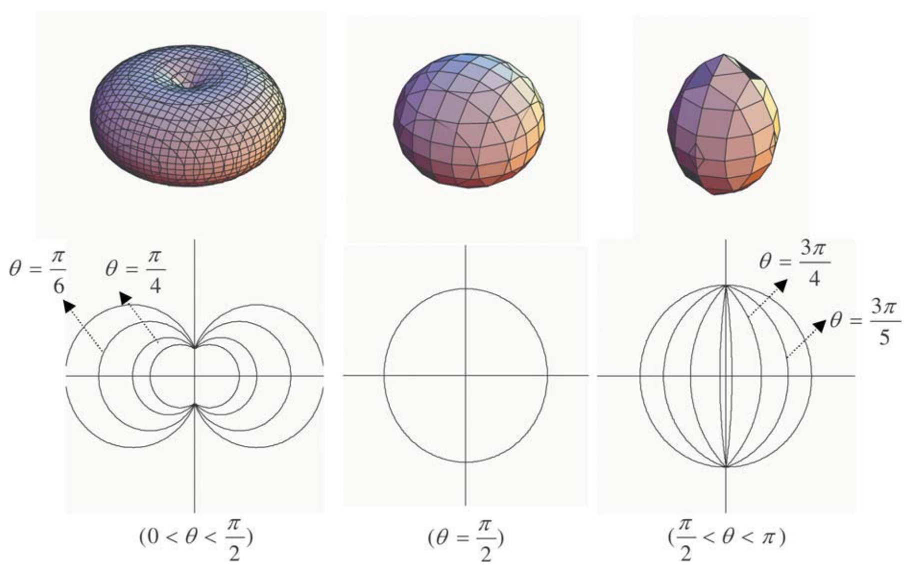

Figure 2. Surfaces of constant θ.

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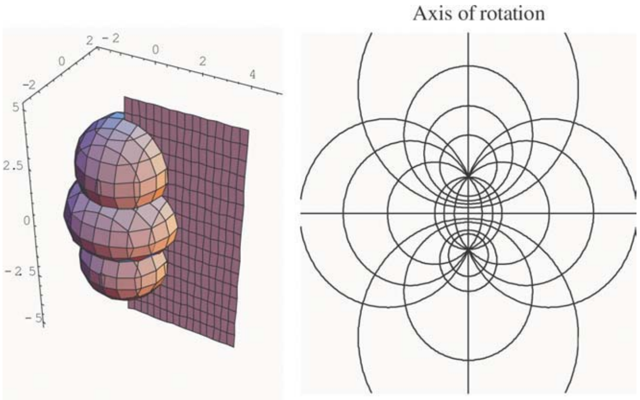

Figure 3. Bispherical coordinates system.

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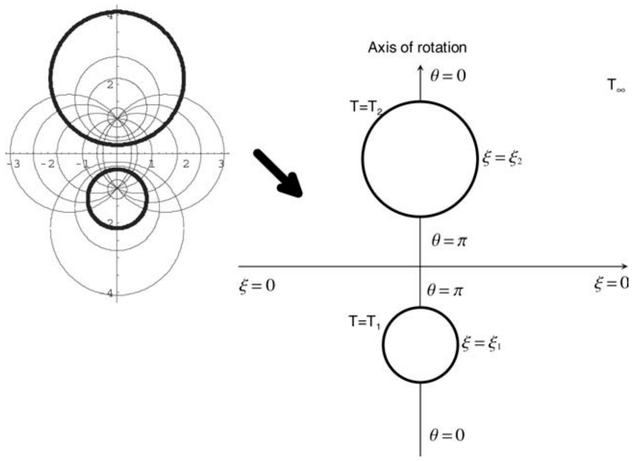

Figure 4. Problem configuration.

The bispherical coordinates system put together is sketched in Figure 3.

It can be shown that specifying the radius of each of the two spheres (r_1 and r_2) and the center-to-center distance (H) fixes a particular bispherical coordinates system in the sense that ξ_1 (first sphere), ξ_2 (second sphere), and a are uniquely determined. It is not difficult to show that

$$\xi_1 = -\sinh^{-1}\frac{a}{r_1}, \xi_2 = \sinh^{-1}\frac{a}{r_2}, \text{and } a = \frac{\sqrt{(H+r_1+r_2)+(H+r_1-r_2)+(H-r_1+r_2)+(H-r_1-r_2)}}{2H} \tag{5}$$

The scale factors for the bispherical coordinates system are

$$h_1 = h_\theta = \frac{a}{\cosh\xi - \cos\theta}, h_2 = h_\xi = \frac{a}{\cosh\xi - \cos\theta}, \text{and } h_3 = h_\gamma = \frac{a\sin\theta}{\cosh\xi - \cos\theta} \tag{6}$$

Heat Conduction Model and Solution

The problem considered here is that of two isothermal spheres, possibly of different diameters and different temperatures, placed at some distance from each other in an infinite fluid. The temperature of the first sphere ($\xi = \xi_1 < 0$) is maintained at T_1, whereas the temperature of the second sphere ($\xi = \xi_2 > 0$) is maintained at T_2. The temperature far away from the two spheres is denoted by T_∞, Figure 4.

The steady states axisymmetric equation of heat conduction in bispherical coordinates can be written as

$$\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cosh\xi - \cos\theta}\frac{\partial T}{\partial\theta}\right) + \frac{\partial}{\partial\xi}\left(\frac{\sin\theta}{\cosh\xi - \cos\theta}\frac{\partial T}{\partial\xi}\right) = 0 \tag{7}$$

Along $\theta = 0$ or $\theta = \pi$, we expect no variation of temperature with respect to the direction θ. We may then write $\frac{\partial T}{\partial\theta}\Big|_{\theta=0} = 0$ and $\frac{\partial T}{\partial\theta}\Big|_{\theta=\pi} = 0$. Far away from the spheres, the temperature is T_∞. It is very important to recognize that the far field is represented in bispherical coordinates by the single point $(\theta, \xi) \to (0, 0)$. This single point represents the "huge sphere" with infinite radius that engulfs the whole domain. The rectangular map of the region for the problem is shown in Figure 5.

It is of well-established traditions in fluid and thermal sciences to express the governing equations in dimensionless forms. Nondimensionalization of equations helps to eliminate several physical constraints such as the use of particular units of measurements. We define the dimensionless temperature φ as

$$\varphi = \frac{T - T_\infty}{T_1 - T_\infty} \tag{8}$$

Accordingly, Eq. 7 can be rewritten in terms of the dimensionless temperature as

$$\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cosh\xi - \cos\theta}\frac{\partial\varphi}{\partial\theta}\right) + \frac{\partial}{\partial\xi}\left(\frac{\sin\theta}{\cosh\xi - \cos\theta}\frac{\partial\varphi}{\partial\xi}\right) = 0 \tag{9}$$

The rectangular region for the problem now looks like Figure 6. Note that $\varphi_2 = \frac{T_2 - T_\infty}{T_1 - T_\infty}$.

The reader may wish to try and then realize that Eq. 9 is not separable in the classical sense. The bispherical coordinates system is R-separable instead.[31] A solution $\psi(x_1,x_2,x_3)$ of a differential equation in three variables is R-separable if it can be written in the form $\psi(x_1,x_2,x_3) =$

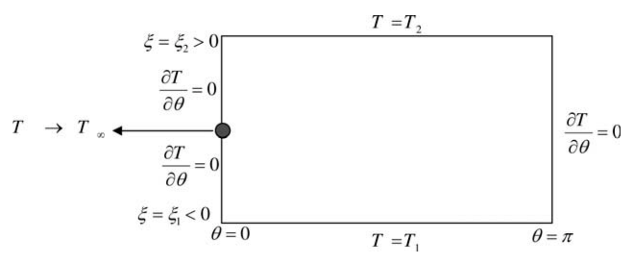

Figure 5. Rectangular region for the problem.

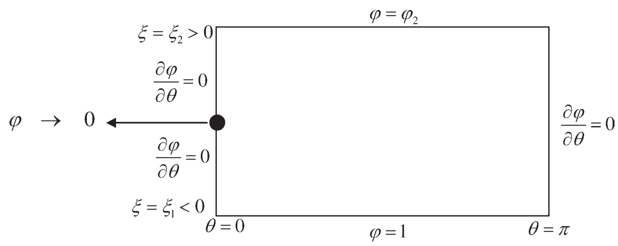

Figure 6. Rectangular region for the problem with dimensionless boundary conditions.

$R(x_1, x_2, x_3)A(x_1)B(x_2)C(x_2)$, where $R(x_1, x_2, x_3)$ contains no factors that are functions of one variable. $R(x_1, x_2, x_3)$ is called the modulation factor because it modifies all factored solutions in the same way.[30]

Equation 9 admits the separation form,

$$\varphi \equiv \sqrt{\cosh\xi - \cos\theta}\; f(\xi)\;\; g(\theta) \tag{10}$$

Substituting the trial solution, Eq. 10 in Eq. 9, one may finally get the following solution

$$\varphi = \sqrt{\cosh\xi - \cos\theta}\sum_{n=0}^{\infty}\left[A_n\, e^{(n+\frac{1}{2})\,\xi} + B_n\, e^{-(n+\frac{1}{2})\,\xi}\right]P_n(\cos\theta) \tag{11}$$

where $P_n(\cos\theta)$ are the Legendre polynomials of the first kind. Equation 11 satisfies the left and right boundaries of Figure 6 and also $\varphi \to 0$ as $(\theta, \xi) \to (0, 0)$. The top and bottom boundaries are used to determine the coefficients A_n and B_n. We will need the following Fourier expansion of $\frac{1}{\sqrt{\cosh\xi - \cos\theta}}$.

$$\frac{1}{\sqrt{\cosh\xi - \cos\theta}} = \sqrt{2}\sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})|\xi|}P_n(\cos\theta) \tag{12}$$

Applying the top and bottom boundary conditions, one ends up with two equations in the unknown coefficients A_n and B_n. Solving the two equations, one finds that

$$A_n = \frac{\sqrt{2}\,(e^{(2n+1)\,\xi_1} - \varphi_2)}{e^{(2n+1)\,\xi_1} - e^{(2n+1)\,\xi_2}},\ \text{and}$$
$$B_n = \frac{\sqrt{2}\,e^{(2n+1)\,\xi_1}(\varphi_2 - e^{(2n+1)\,\xi_2})}{e^{(2n+1)\,\xi_1} - e^{(2n+1)\,\xi_2}} \tag{13}$$

The solution can now be written as,

$$\varphi = \sqrt{\cosh\xi - \cos\theta}\sum_{n=0}^{\infty}\left[\frac{\sqrt{2}\,(e^{(2n+1)\,\xi_1} - \varphi_2)}{e^{(2n+1)\,\xi_1} - e^{(2n+1)\,\xi_2}}\, e^{(n+\frac{1}{2})\,\xi} + \frac{\sqrt{2}\,e^{(2n+1)\,\xi_1}(\varphi_2 - e^{(2n+1)\,\xi_2})}{e^{(2n+1)\,\xi_1} - e^{(2n+1)\,\xi_2}}\, e^{-(n+\frac{1}{2})\,\xi}\right]P_n(\cos\theta) \tag{14}$$

Figure 7 shows the isotherms of a typical solution (the case $r_1 = 1$, $r_2 = 3$, $H = 5$, and $\varphi_2 = 1$).

Rate of heat transfer

The local rate of heat transfer from any of the two spheres is

$$q(\theta) = -k\left(\frac{1}{h_\xi}\frac{\partial T}{\partial \xi}\right)_{\xi=\xi*} \tag{15}$$

Here, $\xi*$ may be either ξ_1 or ξ_2. We define the local Nusselt number as,

$$N_u(\theta) = \frac{2\,r_1\,q(\theta)}{k\,(T_1 - T_\infty)} = -2r_1\left(\frac{1}{h_\xi}\frac{\partial\varphi}{\partial\xi}\right)_{\xi=\xi*} \tag{16}$$

In Eq. 16, we chose to scale by the radius of the lower sphere. We fix the radius of the lower sphere at a value of unity. Its scaled temperature is also fixed at unity. The reader should see that we do not lose generality by fixing such values as the relative sizes of the spheres can be controlled by choosing an appropriate size of the top sphere as to obtain the desired size relative to the fixed-size lower sphere. By the same argument, we number the given two spheres with two different temperatures (rotate the coordinates system up-side-down if necessary) appropriately and use Eq. 16 for scaling. The case when the two spheres are at the same temperature as the far field temperature is trivial, Eq. 16 with $r_1 = 1$ can now be written as,

$$N_u(\theta) = -2\left[\frac{(\cosh\xi - \cos\theta)}{a}\frac{\partial\varphi}{\partial\xi}\right]_{\xi=\xi*} \tag{17}$$

So for the reasons mentioned earlier, we will only calculate the heat transfer coefficient on the top sphere (ξ_2). Using Eq. 14, the following is an explicit expression of the local Nusselt number at the top sphere.

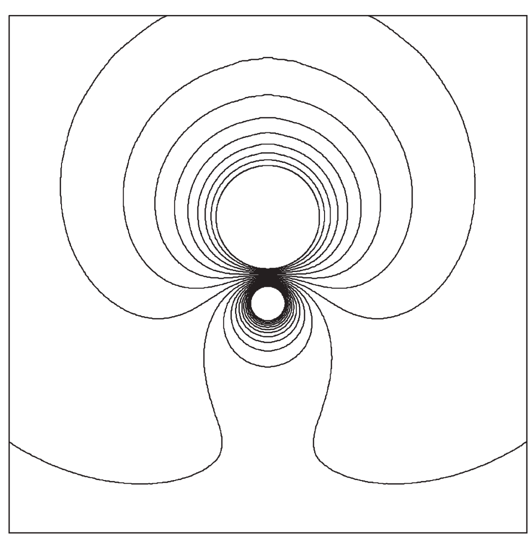

Figure 7. Isotherms of the case $r_1 = 1$, $r_2 = 3$, $H = 5$, and $\varphi_2 = -1$.

$$N_u(\theta) = -\frac{2}{a}\left\{\frac{\sinh\xi_2}{2}(\cosh\xi_2 - \cos\theta)^{\frac{1}{2}}\right.$$
$$\sum_{n=0}^{\infty}\left[A_n\, e^{(n+\frac{1}{2})\,\xi_2} + B_n\, e^{-(n+\frac{1}{2})\,\xi_2}\right]P_n(\cos\theta)$$
$$+ (\cosh\xi_2 - \cos\theta)^{\frac{3}{2}}\sum_{n=0}^{\infty}\left[A_n\left(n+\frac{1}{2}\right)e^{(n+\frac{1}{2})\,\xi_2}\right.$$
$$\left.\left. - \left(n+\frac{1}{2}\right)B_n\, e^{-(n+\frac{1}{2})\,\xi_2}\right]P_n(\cos\theta)\right\} \quad (18)$$

Averaging Nusselt number over the surface of a sphere gives the averaged Nusselt number $\overline{N_u}$.

$$\overline{N_u} = \int_\sigma N_u(\theta)\, dA \Big/ \int_\sigma dA \quad (19)$$

On the top sphere,

$$dA = \frac{a^2\sin\theta}{(\cosh\xi_2 - \cos\theta)^2}d\theta\, d\gamma, \text{and } A = \frac{4\,\pi\, a^2}{\sinh^2\xi_2} \quad (20)$$

Using Eqs. 18 and 20 in Eq. 19 and integrating over θ ($0 \leq \theta \leq \pi$), we easily see that the following integrals are needed.

$$\int_0^\pi (\cosh\xi_2 - \cos\theta)^{-\frac{1}{2}}P_n(\cos\theta)\ \sin\theta\, d\theta \quad (21)$$

$$\int_0^\pi (\cosh\xi_2 - \cos\theta)^{-\frac{3}{2}}P_n(\cos\theta)\ \sin\theta\, d\theta \quad (22)$$

Formulas for the integrals (21) and (22) can be obtained,[32] by noting that

$$\frac{1}{2}\Gamma(1+\mu)\int_{-1}^{1} P_n(z)\,(\omega - z)^{-\mu-1}dz = (\omega^2-1)^{-\mu/2}e^{-i\pi\mu}Q_n^\mu(\omega) \quad (23)$$

where Γ is the gamma function, Q_n^μ is the Legendre polynomials of the second kind. The Legendre polynomials of the second kind are related to the hypergeometric functions (${}_2F_1$) by,

$$Q_n^\mu(\omega) = \frac{e^{\mu\,\pi i}\Gamma(n+\mu+1)\,\Gamma(\frac{1}{2})}{2^{n+1}\,\Gamma(n+\frac{3}{2})}(\omega^2-1)^{\frac{\mu}{2}}\,\omega^{-n-\mu-1} \times {}_2F_1\left(\frac{n+\mu+2}{2}, \frac{n+\mu+1}{2}; n+\frac{3}{2}; \frac{1}{\omega^2}\right) \quad (24)$$

Using the famous transformation formula of the hypergeometric functions

$${}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha}\ \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}\ {}_2F_1\left(\alpha,\gamma-\beta;\alpha-\beta+1;\frac{1}{1-z}\right) + (1-z)^{-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}\ {}_2F_1\left(\beta,\gamma-\alpha;\beta-\alpha+1;\frac{1}{1-z}\right) \quad (25)$$

shows that these functions are, in our case, terminating and not infinite.[32] Some quite tedious work leads to the following very simple formulas for the integrals (21) and (22).

$$\int_0^\pi (\cosh\xi_2 - \cos\theta)^{-\frac{1}{2}}P_n(\cos\theta)\ \sin\theta\, d\theta = \frac{2\sqrt{2}}{2n+1}e^{-(n+1/2)\,\xi_2} \quad (26)$$

$$\int_0^\pi (\cosh\xi_2 - \cos\theta)^{-\frac{3}{2}}P_n(\cos\theta)\ \sin\theta\, d\theta = \frac{2\sqrt{2}}{\sinh\xi_2}e^{-(n+1/2)\,\xi_2} \quad (27)$$

The average Nusselt number can now be written as,

$$\overline{N_u} = -\frac{2\sqrt{2}\ \sinh^2\xi_2}{a}\sum_{n=0}^{\infty}A_n \quad (28)$$

which can be explicitly rewritten using Eq. 13 as,

$$\overline{N_u} = -\frac{4\ \sinh^2\xi_2}{a}\sum_{n=0}^{\infty}\frac{e^{(2n+1)\,\xi_1} - \varphi_2}{e^{(2n+1)\,\xi_1} - e^{(2n+1)\,\xi_2}} \quad (29)$$

Verification of solution

As the distance between the two spheres increases, the effect of the existence of one sphere on the other becomes negligible. Consider, for example, two spheres having the same diameters. Consider, for simplicity, that the temperature of the top sphere is also unity. Equation 14 reduces to

$$\varphi = \sqrt{2(\cosh\xi - \cos\theta)}\ \sum_{n=0}^{\infty} e^{-(n+1/2)\,\xi}\,\frac{(1+e^{(2n+1)\,\xi})}{(1+e^{(2n+1)\,\xi_2})} \times P_n(\cos\theta) \quad (30)$$

Figure 8 shows the isotherms for this case of the two spheres of the same size and at the same temperature. The

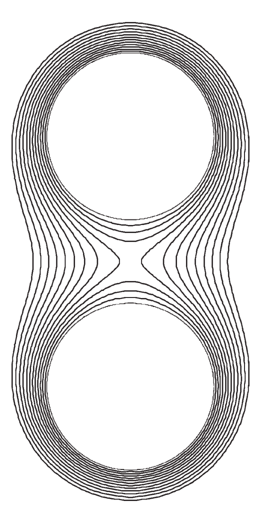

Figure 8. Isotherms of the case $r_1 = r_2 = 1$ and $\varphi_2 = 1$.

temperature gradients are obviously lower between the two spheres ($\theta = \pi$) than at the two far edges ($\theta = 0$).

Expression (18) which gives the local rate of heat transfer from the top sphere takes the following explicit form.

$$N_u(\theta) = -2\sqrt{2(\cosh\xi - \cos\theta)} \times \sum_{n=0}^{\infty} e^{-(n+1/2)\xi} \frac{-1 - n + e^{\xi_2}(-e^{\xi_2} + e^{2n\xi_2})n + e^{(3+2n)\xi_2}(1+n) + (e^{\xi_2} - e^{2(n+1)\xi_2})(1+2n)\cos\theta}{(-1 + e^{2\xi_2})(1 + e^{(2n+1)\xi_2})} P_n(\cos\theta) \quad (31)$$

With a careful investigation of the little "unpleasant" expression, we find that as $\xi_2 \to \infty$ (i.e., the two spheres get far away from each other) the only term that contributes to the limit is that with $n = 0$. The limiting value of $N_u(\theta)$ is -2. Note that the sphere is hotter than the surrounding medium and the negative sign is because of the fact that the direction of increasing ξ is toward the inside of the top sphere. Examination of the expression for the average Nusselt number in Eq. 29, one can also find out that as $\xi_2 \to \infty$, $\overline{N_u} \to -2$ with, again, only the term with $n = 0$ contributing to the limit value. Figure 9 shows the variation of N_u along the surface ξ_2 for the case under consideration. As H increases, N_u becomes more uniform along the surface. As the two spheres get far apart, the existence of one is not felt by the other. It is not surprising that one should be able to obtain the same value of N_u if the problem of a single sphere was considered in spherical coordinates. The value obtained by analyzing the problem of a single sphere in spherical coordinates is 2.

Truncation error

We use Eq. 30 to estimate the error that results from considering only the first few terms (say N) to calculate the sum of the series solution. Because of symmetry, we consider the semi-infinite space $\xi > 0$. We can write,

$$\text{Error} = \left| \sum_{n=N+1}^{\infty} \sqrt{2(\cosh\xi - \cos\theta)}\, e^{-(n+1/2)\xi} \frac{(1 + e^{(2n+1)\xi})}{(1 + e^{(2n+1)\xi_2})} \times P_n(\cos\theta) \right| \le \sum_{n=N+1}^{\infty} \sqrt{2(\cosh\xi - \cos\theta)}\, e^{-(n+1/2)\xi} \quad (32)$$

as Legendre polynomials are bounded by 1 and $\frac{(1+e^{(2n+1)\xi})}{(1+e^{(2n+1)\xi_2})} \le 1$ as $\xi \le \xi_2$.

As $\sqrt{2(\cosh\xi - \cos\theta)} \le 2\sqrt{\cosh\xi} = \sqrt{2(e^{\xi} + e^{-\xi})} \le 2\sqrt{e^{\xi}} = 2\,e^{\xi/2}$, we can write

$$\text{Error} \le \sum_{n=N+1}^{\infty} \sqrt{2(\cosh\xi - \cos\theta)}\, e^{-(n+1/2)\xi} \le \sum_{n=N+1}^{\infty} 2\, e^{-n\xi} = 2\frac{e^{-N\xi}}{e^{\xi} - 1} \le 2\, e^{-(N+1)\xi} \quad (33)$$

as the sum $\sum_{n=N+1}^{\infty} 2\, e^{-n\xi}$ is geometric and has the indicated value in (Eq. 33). Thus, the error decays exponentially. The results presented in this article are obtained using $N = 45$.

The parameters

We consider here some cases as to understand how the heat transfer coefficient changes with the sizes of the spheres, their temperatures, and the gap between them. All values of Nusselt number refer to the top sphere.

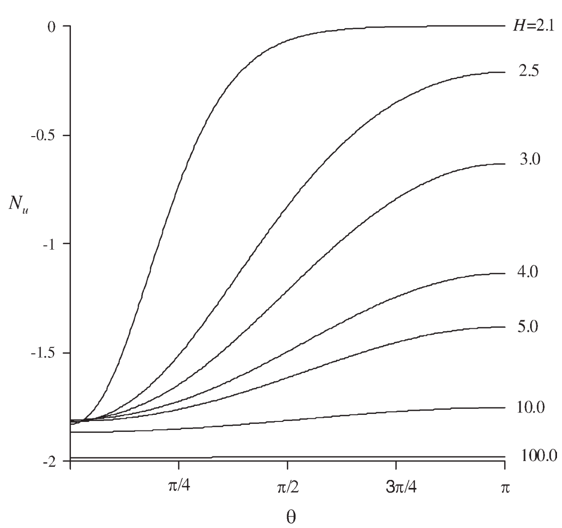

Figure 9. Variation of N_u along the surface (ξ_2) for the case $r_1 = r_2 = 1$ and $\varphi_2 = 1$.

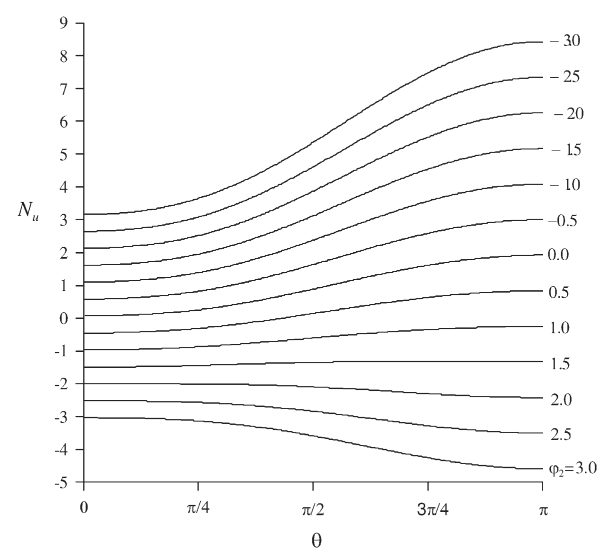

Figure 10. Variation of N_u along the surface (ξ_2) for the case $r_1 = 1$, $r_2 = 2$, and $H = 4$.

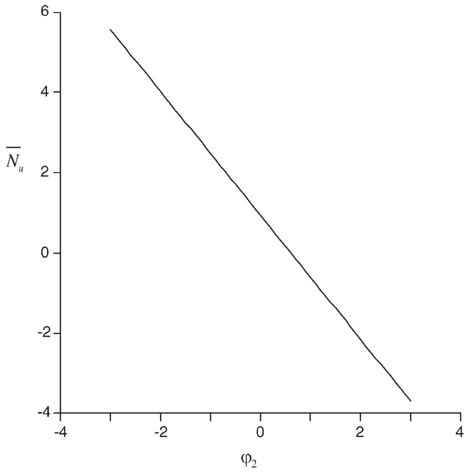

Figure 11. Variation of $\overline{N_u}$ for the case r_1 = 1, r_2 = 2, and H = 4.

Consider the case when $r_2 = 2$ (remember we fix $r_1 = 1$) and $H = 4$. Figure 10 shows the variation of N_u along the surface of the top sphere for different values of φ_2 ($\varphi_1 = 1$). Because of the direction of increasing ξ in the bispherical coordinates (toward the inside of the top sphere), positive N_u means that there is a transfer of heat to the sphere whereas negative values indicate that heat is transferred from the sphere to the surroundings. As expected, large thermal gradients exist in the region between the two spheres (near $\theta = \pi$) when the difference between the temperatures of the two spheres is large when compared with the region on the other edges of the spheres ($\theta = 0$). Negative values of φ_2 ($\varphi = (T - T_\infty)/(T_1 - T_\infty)$) indicate that the top sphere is at temperature lower than the far field; whereas the temperature of the lower sphere is at higher temperature than the far field or vice versa.

Figure 11 shows the average Nusselt number for the cases under consideration. The relation between $\overline{N_u}$ and φ_2 is linear as clearly seen in Eq. 29. The isotherms for some cases are shown in Figure 12. Notice that for some cases such as that when $\varphi_2 = 0.5$, heat is transferred to the sphere through some part of the surface; whereas heat is transferred from the sphere to the surroundings through the remaining part of the surface of the sphere. In other cases, heat is completely transferred from the sphere to the surroundings ($\varphi_2 = 3$) or transferred completely from the surroundings to the sphere ($\varphi_2 = -3$).

Figure 13 confirms the fact that the heat transfer coefficient approaches a constant value as the two spheres get far away from each other. The figure shows the variation of the averaged Nusselt number with the center-to-center distance for the case $r_1 = 1$, $r_2 = 1/2$, and $\varphi_2 = 1/3$. As the distance increases, the existence of one sphere does not affect the other and the averaged Nusselt number approaches a constant value. The approached value is determined by the fact that we used the diameter of the bottom sphere and the temperature difference between the bottom sphere and the far field for scaling the heat transfer coefficient (Nusselt number). It is interesting to observe how $\overline{N_u}$ changes sign as the distance between the two spheres increases. When the distance is small, the temperature of the bottom sphere is larger and heat is transferred to the top sphere ($\overline{N_u} > 0$). As the

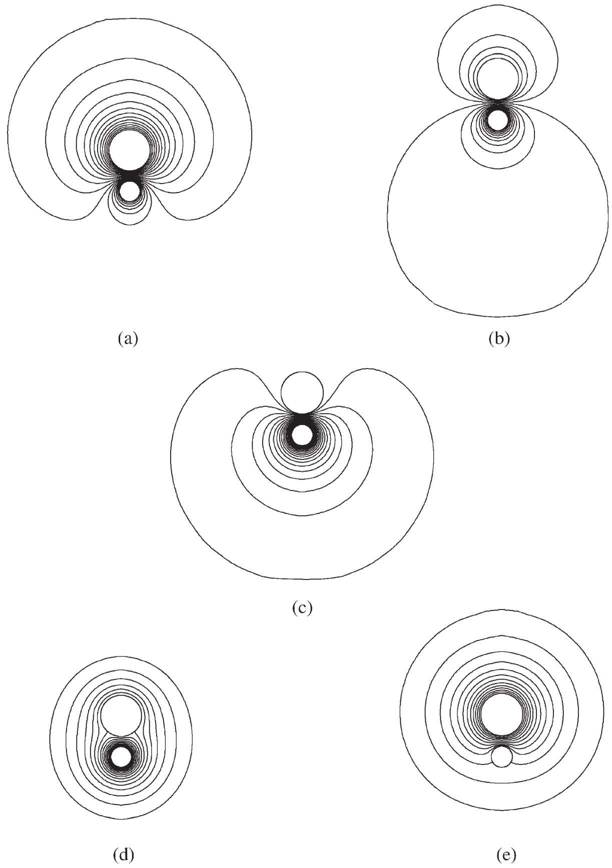

Figure 12. Isotherms for the case r_1 = 1, r_2 = 2, and H = 1.

(a) $\varphi_2 = -3.0$, (b) $\varphi_2 = -0.5$, (c) $\varphi_2 = 0.0$, (d) $\varphi_2 = 0.5$, (e) $\varphi_2 = 3.0$

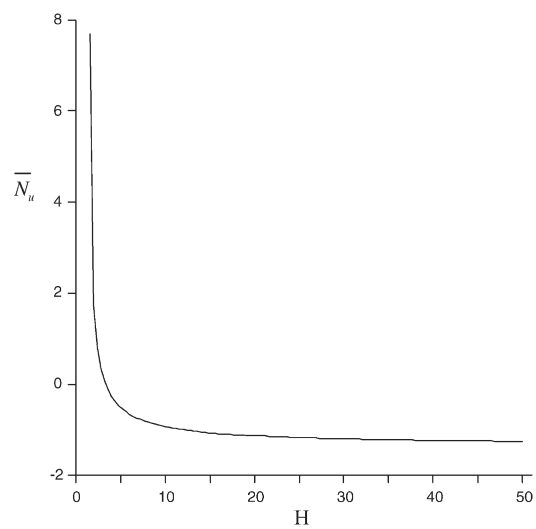

Figure 13. Variation of $\overline{N_u}$ for the case r_1 = 1, r_2 = 1/2, and φ_2 = 1/3.

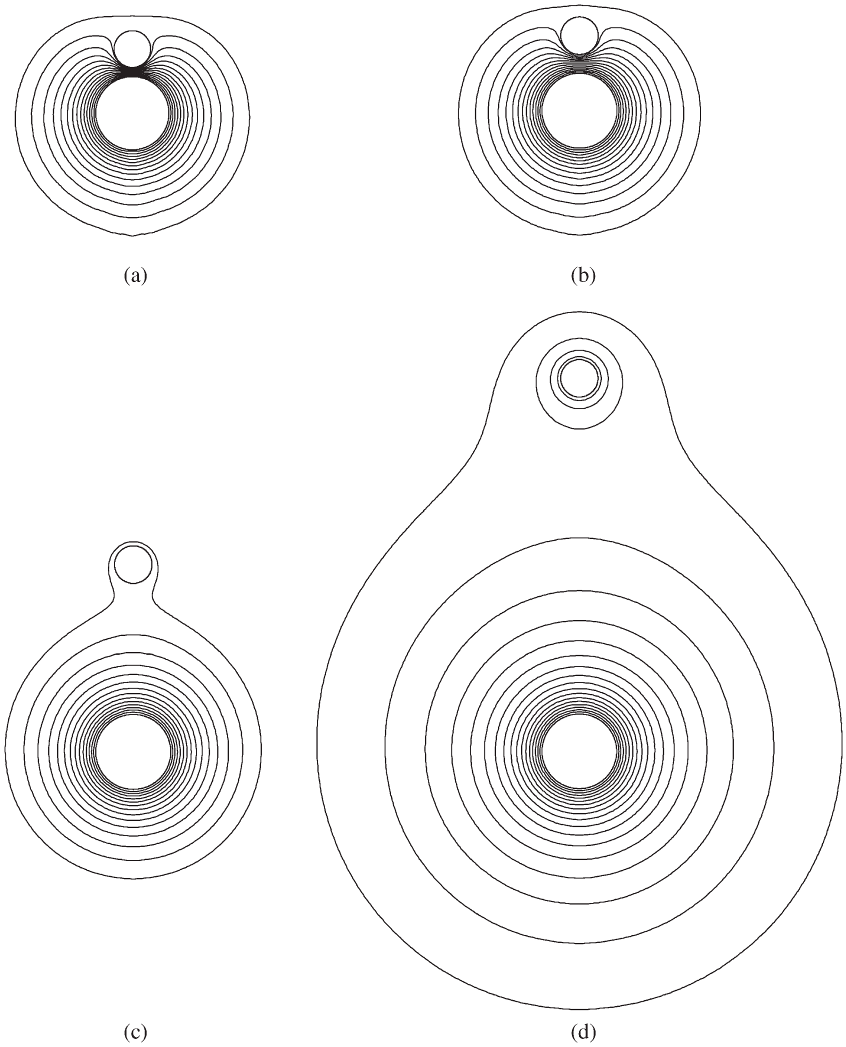

Figure 14. Isotherms for the case $r_1 = 1$, $r_2 = 1/2$, and $\varphi_2 = 1/3$.

(a) $H = 7/4$, (b) $H = 2$, (c) $H = 5$, and (d) $H = 10$

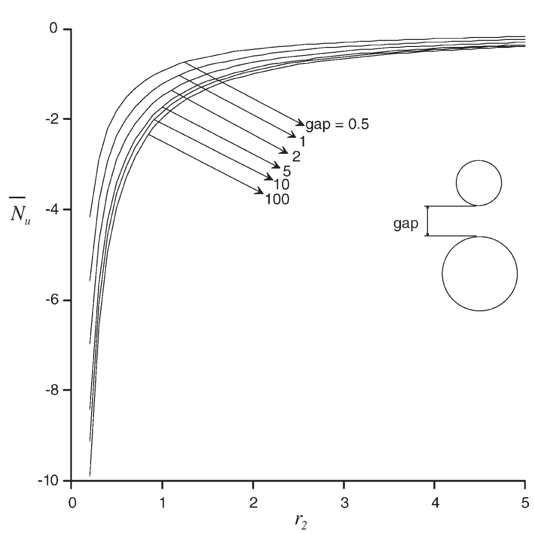

Figure 15. Variation of $\overline{N_u}$ with r_2 for the case $r_1 = 1$ and $\varphi_2 = 1$.

distance increases, the top sphere gets free of the influence of the bottom sphere. As the temperature of the top sphere is still larger than the far field, heat is transferred from the top sphere to the surroundings ($\overline{N_u}<0$). Figure 14 shows some isotherms of this case.

Figure 15 shows the averaged Nusselt number variation with the radius of the top sphere for a fixed gap size. The gap is the distance from the surface of one sphere to the surface of the other along the center-to-center line. As the diameter of the top sphere becomes large, the impact of the gap on the heat transfer rate subsides. This is expected as the gap size relative to the size of the sphere becomes small.

Conclusions

An exact simple solution of the problem of heat conduction from two spheres, possibly of different diameters and different temperatures, placed in an infinite fluid at some distance from each other is given in Eq. 14. The truncation error of the series solution obtained in this article decays exponentially. The explicit expressions of the local and average Nusselt numbers are given respectively, by Eqs. 18 and 29. The results of this study are verified by comparing the value of the rate of heat transfer in the case when the distance between the two spheres becomes very large to the value obtained by analyzing the problem of heat transfer from a single sphere using spherical coordinates.

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